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Geometrical treatment of systems driven by coloured noise

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Abstract. We use geometrical methods and functional derivative techniques to analyse the stochastic behaviour of multidegree-of-freedom nonlinear systems driven by Gaussian noises with arbitrary correlation functions. The method has the advantage of being also applicable to systems with an infinite number of degrees of freedom (partial differential equations) with some kind of Hamiltonian structure, and gives some insight into the nature of the approximations that lead to some of the main effective Fokker-Planck equations used in the literature.

1. Introduction

The study of the properties of nonlinear systems driven by stochastic forces is a topic of current interest. There is a large class of physical situations in which the stochastic nature of some of the relevant parameters involved in the problem play a prominent role in the evolution of the system [1-4]. Of particular significance is the analysis of the behaviour of systems perturbed by coloured noise, i.e., with a non-zero correlation time. It is now well established that the consideration of the finite correlation time of the noise may be of importance in the description of a number of physical problems, e.g., statistical properties of dye lasers [5-7], noise-induced phase transitions in chemically reacting systems [2, 8, 9], optical bistability [10], and liquid crystals [11]. Several techniques have been applied to obtain approximate Fokker-Planck equations to describe the behaviour of the probability density of the system [2, 4, 12-16]. Although the relation among the different approximate Fokker-Planck equations is now becoming clear [17, 18], it would be certainly of interest to have a simple, powerful, unifying formalism to deal with a broad class of systems under the influence of coloured noise and that allows one to identify some of the approximations that lead to the different Fokker-Planck equations.

Quite recently a geometrical treatment of stochastic systems driven by white noise has been developed [19, 20]. This approach allows the use of a geometrical (i.e., intrinsic) language independent of the coordinates and provides general advantages that can contribute towards clarifying the analysis of stochastic systems, e.g., the derivation of the Fokker-Planck equations is simplified and the nature of some of the approximations appears clearer, the formalism is valid for multidimensional systems and also for a broad class of stochastic partial differential equations, and the invariance of the effective Fokker-Planck equations (EFPE) under transformations is very easy to recognise.

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In this paper we use functional derivative techniques [12, 21–23] to extend those results to systems perturbed by coloured noise with arbitrary correlation functions. In section 2 we re-obtain two of the main EFPE used in the literature. We start with the Langevin equation to arrive at an exact evolution equation for the mean value $\langle A_t \rangle$ of a dynamical function, in which a stochastic field $\boldsymbol{D}$ appears. In order to obtain a Fokker–Planck type equation it is necessary to replace this stochastic field by a deterministic one. We examine the approximations that lead to the best Fokker–Planck equation (BFPE) [12, 13] and to the Fox equation [14], that are here obtained for multidimensional systems perturbed by an arbitrary number of Gaussian correlated noises. As a by-product of the derivation we also provide a necessary and sufficient condition to obtain an exact Fokker–Planck equation for the system, which is the multidimensional extension of that found by Faetti and Grigolini [17]. It should be, however, emphasised that our aim in this section is to illustrate the usefulness of the geometrical approach and not to discuss the range of applicability of the different EFPE, a problem that has already been considered in the literature [4, 7, 24–28]. Section 3 is devoted to the discussion of two simple examples, included mainly for pedagogical reasons: linear systems, deriving a closed equation for the first moment of the position vector, and one-dimensional nonlinear systems, for which we obtain a compact form of the diffusion function similar to that obtained recently by van Kampen [29]. In section 4 we apply the formalism to classical Hamiltonian systems, giving the expression of the BFPE and Fox equation in terms of Poisson brackets. Newtonian systems and two very important partial differential equations, the nonlinear Klein–Gordon equation and the Korteweg–de Vries equation, are discussed as examples, obtaining some qualitative properties for the mean value of relevant dynamical variables. In section 5 we apply the BFPE to quantum systems perturbed by classical noise. We treat in detail the relaxation of an arbitrary spin in the presence of a fluctuating magnetic field, previously considered by Itzykson [30] in order to explain the anomalous magnetic moment, and by Faid and Fox [31] in their generalisation of Kubo's theory of spectroscopic line shapes. Finally, in section 6 we summarise our main conclusions.

2. Geometrical derivation of Fokker–Planck equations

We consider the stochastic differential equation (summation over repeated indices is implied)

$$\dot{x} = \boldsymbol{F}(x) + \xi_k(t)\boldsymbol{G}^k(x) \tag{1}$$

where x is a point of a manifold M, $\boldsymbol{F}$ and $\boldsymbol{G}^k$ are vector fields on M and $\xi_k(t)$ are Gaussian coloured noises with zero mean value and arbitrary correlation function

$$\langle \xi_k(t) \rangle = 0 \tag{2}$$

$$\langle \xi_k(t)\xi_{k'}(t') \rangle = \Gamma_{kk'}(t-t'). \tag{3}$$

The equation (1) defines a stochastic flow $\phi^{(t,t')}: M \to M$. The smooth dynamical function $A: M \to \mathbb{R}$ evolves in the usual way

$$A_t \equiv A \circ \phi^{(t,0)} \tag{4}$$

following the equation

$$\dot{A}_t = L_{\boldsymbol{F}_t + \xi_k(t)\boldsymbol{G}_t^k}(A_t) \tag{5}$$

where L_{F_t} is the Lie derivative [32] in the direction of the field $F_t \equiv [\phi^{(t,0)}]^*(F)$. Averaging (4) over the realisations of the noises ξ_k and recalling that $p(x, t|y, 0) = \langle\delta(x - \phi^{(t,0)}y)\rangle$ we have,

$$\langle A_t\rangle(y) = \int_M \mathrm{d}x\, p(x, t|y, 0)A(x). \tag{6}$$

We will derive the evolution equation for the mean value $\langle A_t\rangle$ and we will use it to arrive at an equation for the probability density. In order to do this we average (5) and apply the well known properties of the Lie derivative to get

$$\langle \dot{A}_t\rangle = \langle[L_F(A)]_t\rangle + \langle\xi_k(t)[L_{G^k}A]_t\rangle. \tag{7}$$

In the appendix we obtain an expression for $\langle\xi_k(t)B_t\rangle$, B_t being an arbitrary dynamical variable. Using this result we finally get

$$\langle \dot{A}_t\rangle = \langle[L_F A]_t\rangle + \langle[L_{D^k(t)}L_{G^k}(A)]_t\rangle \tag{8}$$

where the field $D^k(t)$ contains the memory effects due to the non-zero correlation time of the noises and is given by

$$D^k(t) = \sum_{k'}\int_0^t \mathrm{d}s\, \Gamma_{kk'}(s)[\phi^{(t-s,t)}]^* G^{k'}. \tag{9}$$

This is an exact formal expression although not very useful since the field D^k is, in general, stochastic. This fact prevents the use of (6) to obtain a Fokker–Planck type equation for the conditional probability. The approximations that yield an equation of that type consist of the replacement of the exact $D^k(t)$ by a deterministic field. If we perform such an approximation, using (6) and integrating by parts, we obtain

$$\frac{\partial}{\partial t} p(x, t|y, 0)\, \mathrm{d}x = \mathscr{L}[p(x, t|y, 0)\, \mathrm{d}x] \tag{10}$$

with

$$\mathscr{L} = -L_F + L_{G^k}L_{D^k(t)}. \tag{11}$$

The best Fokker–Planck approximation [12, 13] (BFPA) takes the form

$$D^k(t) = \sum_{k'}\int_0^t \mathrm{d}s\, \Gamma_{kk'}(s)[\phi_0^{-s}]^* G^{k'} \tag{12}$$

where ϕ_0^s is the deterministic flow. It will be valid if the deterministic flow is close to the stochastic one in the integral domain. This is a qualitative argument which can provide validity criteria for each problem depending not only on the characteristic parameters of the noises but also on the vector fields F and G. As stated in the introduction it is not the aim of this paper to discuss the general range of applicability of the BFPE, a topic that has been treated in a number of recent studies [26, 27, 33].

The more familiar expression of the diffusion field D^k in terms of a correlation time expansion is obtained by recalling that

$$[\phi_0^s]^* = \exp(sL_F). \tag{13}$$

If, moreover, we now assume that the stochastic forces are of the Ornstein–Uhlenbeck (OU) type [2] with correlation function

$$\Gamma_{kk'}(s) = \frac{\sigma_{kk'}^2}{2\tau_{kk'}}\exp\left(-\frac{s}{\tau_{kk'}}\right) \tag{14}$$

and neglect the transient terms, we get

$$\boldsymbol{D}^k = \sum_{k'} \frac{\sigma^2_{kk'}}{2} \sum_{n=0}^{\infty} (-\tau_{kk'} L_F)^n \boldsymbol{G}^{k'} \tag{15}$$

or formally

$$\boldsymbol{D}^k = \sum_{k'} \frac{\sigma^2_{kk'}}{2} [\mathbf{1} + \tau_{kk'} L_F]^{-1} \boldsymbol{G}^{k'}. \tag{16}$$

Another widely used EFPE is the one proposed by Fox [14]. This equation is obtained by making a linearisation of the deterministic flow $[\phi_0^{-s}]^*$ at each point x of the manifold and it is also known as local linearisation [17, 34]. If we can find tensorial fields $\boldsymbol{\Lambda}^k(x)$ such that

$$L_F \boldsymbol{G}^k = \boldsymbol{\Lambda}^k \boldsymbol{G}^k \tag{17}$$

$$L_F \boldsymbol{\Lambda}^k \simeq 0 \tag{18}$$

then we have

$$[\phi_0^s]^* \boldsymbol{G}^k(x) \simeq \exp(s\boldsymbol{\Lambda}^k(x)) \boldsymbol{G}^k(x). \tag{19}$$

Setting (19) in (12) we get the Fox approximation [14] of the fields $\boldsymbol{D}^k$, which reads, under the same conditions of (15) and (16),

$$\boldsymbol{D}^k = \sum_{k'} \frac{\sigma^2_{kk'}}{2} [\mathbf{1} + \tau_{kk'} \boldsymbol{\Lambda}^{k'}]^{-1} \boldsymbol{G}^{k'} \tag{20}$$

where $\mathbf{1}$ is the identity matrix. The transient terms can be neglected only if the matrices $\mathbf{1} + \tau_{kk'} \boldsymbol{\Lambda}^{k'}$ are positive definite and non-singular. These fields inserted in (10) and (11) give the multidimensional version of the equation derived by Fox in [14] for one-dimensional systems perturbed by coloured noise. In this latter case the matrix $\boldsymbol{\Lambda}(x)$ is a scalar function and is univocally determined by (17)

$$\boldsymbol{\Lambda}(x) = \frac{fg' - f'g}{g} = -g\left(\frac{f}{g}\right)'. \tag{21}$$

In the multidimensional case, the condition (17) does not determine the matrix $\boldsymbol{\Lambda}^k(x)$ and (18) must be used to choose in each case the best option.

We finally wish to indicate that there are certain situations in which some of these approximations become exact. If the vector fields $\boldsymbol{G}$ and $\boldsymbol{F}$ satisfy

$$L_G (L_F)^n \boldsymbol{G} = 0 \tag{22}$$

for all n, then the field $\boldsymbol{D}$ is deterministic and the BFPE is exact. A more restrictive but more easily verifiable condition is

$$L_F \boldsymbol{G} = \lambda \boldsymbol{G} \tag{23}$$

and in this case both the BFPE and the Fox equation presented above coincide and are exact. Both conditions (22) and (23) are equivalent for a one-dimension system, and the second reads

$$fg' - gf' = \lambda g \tag{24}$$

which is the same as that found by Faetti and Grigolini [17]. A broad class of systems satisfying (24) is given by

$$\dot{x} = x + \alpha x^{\lambda+1} + \xi(t) x^{\lambda+1} \tag{25}$$

and the corresponding BFPE is (10), (11) with

$$\boldsymbol{D}(t) = \boldsymbol{d}(x, t) = g(x) \int_0^t \mathrm{d}s\, \Gamma(s) \exp\left[-\frac{s}{\tau}(1+\lambda\tau)\right]. \tag{26}$$

We would finally like to point out that for the multidimensional case it is conceivable to find systems that satisfy only the first condition (22). In these cases the BFPE is exact whereas the Fox equation might not be.

3. Simple applications

3.1. Linear systems

We consider the linear equation

$$\dot{x} = Ax + \xi(t) Bx \tag{27}$$

where x is a point of $\mathbb{R}^n$ and A, B are $n \times n$ matrices. In this example it is possible to calculate the BFPA of $\boldsymbol{D}(x)$ which is a linear field given by the matrix D:

$$D = \int_0^t \mathrm{d}s\, \Gamma(s)\, \mathrm{e}^{sA} B\, \mathrm{e}^{-sA}. \tag{28}$$

Assuming that A is a diagonalisable matrix, in the basis that diagonalises A the matrix D is, in the OU case,

$$D_{ij} = \frac{\sigma^2}{2} \frac{B_{ij}}{1 + \tau(a_j - a_i)} \tag{29}$$

where we have assumed that $1/\tau > \mathrm{Re}(a_i - a_j)$ to avoid divergences. We can now apply (8) to obtain a closed equation for the mean value $\langle x_t \rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x_t \rangle = [A + DB] \langle x_t \rangle. \tag{30}$$

3.2. One-dimensional systems

As another example we concentrate our attention on single variable dynamical systems. The equation of motion now is

$$\dot{x} = f(x) + \xi(t) g(x). \tag{31}$$

Applying (12) it is easy to get an expression for the diffusion term of the BFPE which coincides with the usual one in the OU case. The flow defined by (31) is

$$\phi_0^t x = \mathcal{F}^{-1}[\mathcal{F}(x) + t] \tag{32}$$

$\mathcal{F}$ being a primitive of $1/f(x)$. Using this, we get

$$\begin{aligned} D(t; x) &= \int_0^t \mathrm{d}s\, \Gamma(s) \left[\frac{\mathrm{d}\phi_0^s(y)}{\mathrm{d}y}\right]_{y=\phi_0^{-s}x} g(\phi_0^{-s} x) \\ &= \int_0^t \mathrm{d}s\, \Gamma(s) \frac{f(x)}{f(\phi_0^{-s} x)} g(\phi_0^{-s} x). \end{aligned} \tag{33}$$

A similar expression has been recently derived by van Kampen [29], for the additive noise case, i.e., $g(x)=\text{constant}$, using a cumulant expansion. If we now make the change of variable $y=\phi_0^{-s}x$, we obtain, after neglecting transient terms,

$$D(x)=f(x)\int_{x_{st}}^{x}\mathrm{d}y\,\Gamma\left(\int_x^y\frac{\mathrm{d}z}{f(z)}\right)\frac{g(y)}{f^2(y)} \tag{34}$$

where $x_{st}\equiv\lim_{t\to-\infty}\phi_0^t x$ is a stationary point of the flow ϕ_0^t. The change is valid for all y that is not a stationary point, i.e. $f(y)\neq 0$. At these points the function $D(x)$ is given by

$$D(x)=g(x)\int_0^{\infty}\mathrm{d}s\,\Gamma(s)\,\mathrm{e}^{sf'(x)}. \tag{35}$$

Note that the argument of the correlation function Γ in (34) is the time which the deterministic system takes to go from x to y and it is always positive. It is straightforward to check that (34), in the OU case, is the formal solution of the differential equation obtained in [12] for the function $D(x)$.

4. Hamiltonian systems

The general geometric form of the evolution equation derived in section 2 has a simple application in simplectic geometry [35]. We consider a system with the stochastic Hamiltonian

$$H(x)=H_0(x)+\xi_k(t)V_k(x). \tag{36}$$

The symplectic structure associates a vector field $\boldsymbol{X}_H$ with every smooth dynamical function H. The Lie derivative in the direction of such a field acting on functions and Hamiltonian vector fields is given by the Poisson bracket

$$L_{\boldsymbol{X}_H}(V)=\{V,H\} \tag{37}$$

$$L_{\boldsymbol{X}_H}(\boldsymbol{X}_V)=\boldsymbol{X}_{\{V,H\}}. \tag{38}$$

Introducing (37) and (38) in (8), we arrive at

$$\langle\dot{A}_t\rangle=\langle\{A,H_0\}_t\rangle+\langle\{\{A,V_k\},W_k(t)\}_t\rangle \tag{39}$$

where

$$W_k(x;t)=\sum_{k'}\int_0^t\mathrm{d}s\,\Gamma_{kk'}(s)\,V_{k'}(\phi^{(t-s,t)}x). \tag{40}$$

In the BFPA

$$W_k(x;t)=\sum_{k'}\int_0^t\mathrm{d}s\,\Gamma_{kk'}(s)\,V_{k'}(\phi_0^{-s}x). \tag{41}$$

In the OU case this magnitude can be expressed as a τ expansion

$$W_k=\sum_{k'}\frac{\sigma_{kk'}^2}{2}\sum_{n=0}^{\infty}(\tau_{kk'})^n(\mathrm{ad}_{H_0})^n(V_{k'}) \tag{42}$$

with

$$\mathrm{ad}_H(V)\equiv\{H,V\}. \tag{43}$$

The probability measure $p(x, t)\,dx$, where dx is the Liouville measure associated with the symplectic structure, verifies (10). The Lie derivative acting on a volume form $p(x)\,dx$ is

$$L_{X_H}(p(x)\,dx) = (\{p, H\} + p \operatorname{div} \boldsymbol{X}_H)\,dx$$
$$= \{p, H\}\,dx \tag{44}$$

where we have applied the Liouville theorem to the Hamiltonian vector field $\boldsymbol{X}_H$. Therefore, we can write the BFPE in the form

$$\frac{\partial}{\partial t} p(x, t) = -\{p, H_0\} + \{\{p, W_k\}, V_k\}. \tag{45}$$

A detailed analysis of this equation, in the white noise case, is performed in [20], where geometrical properties and optimisation problems are studied. In the following subsections we particularise (39) to different Hamiltonian systems.

The Fox approximation (20) can also be written in terms of Poisson brackets. It yields the same equations (39) and (45) with a different expression for W_k. Let $\lambda_k(x)$ be a dynamical function such that

$$\{V_k, H_0\} = \lambda_k V_k. \tag{46}$$

If we now neglect $\{\lambda_k, H_0\}$ then we have

$$V_k(\phi_0^s x) = \exp[s\lambda_k(x)]\,V_k(x) \tag{47}$$

and, finally,

$$W_k = \sum_{k'} \frac{\sigma_{kk'}^2}{2} \frac{V_{k'}}{1 + \tau_{kk'}\lambda_{k'}} \tag{48}$$

with

$$\lambda_k = \frac{\{V_k, H_0\}}{V_k}. \tag{49}$$

4.1. *Newtonian systems*

We first consider finite dimensional systems with Hamiltonian of the form

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{i=1}^{n} \frac{p_i^2}{2} + V_0(q_1, \ldots, q_n) + \xi(t) V(q_1, \ldots, q_n). \tag{50}$$

The Langevin equation for the coordinates is

$$\ddot{\boldsymbol{q}} = -\operatorname{grad} V_0 - \xi(t) \operatorname{grad} V. \tag{51}$$

In order to show the differences between the white and coloured perturbations, we calculate the BFPA of W up to second order in the correlation time for the OU case

$$W = \frac{\sigma^2}{2}\{V - \tau(\operatorname{grad} V)\cdot\boldsymbol{p} + \tau^2[\boldsymbol{p}\cdot[\operatorname{Hess}(V)\boldsymbol{p}] - (\operatorname{grad} V)\cdot(\operatorname{grad} V_0)]\}. \tag{52}$$

The mean values of $\boldsymbol{q}$ and $\boldsymbol{p}$ evolve following the equation

$$\langle\ddot{\boldsymbol{q}}\rangle = -\langle\operatorname{grad} V_{\mathrm{ef}}\rangle - \langle\Gamma^2(\boldsymbol{q})\boldsymbol{p}\rangle \tag{53}$$

with

$$V_{\mathrm{ef}} = V_0 - \frac{\tau\sigma^2}{4}|\mathrm{grad}\ V|^2 \tag{54}$$

and

$$\Gamma = \tau\sigma\ \mathrm{Hess}(V). \tag{55}$$

Note that if V is a quadratic function the second term in (53) is a dissipative contribution entirely due to the colour of the noise and appears in the second order of τ. For the harmonic oscillator, W can be obtained explicitly in all orders of τ and (53) becomes a closed equation [13], since this a particular case of the linear systems discussed in section 3.1.

Applying (39) to the deterministic energy H_0 and angular momentum $L_{ij} = q_i p_j - q_j p_i$ it is possible to find some remarkable properties of the evolution of such quantities when the perturbation is a white noise:

$$\langle \dot{H}_0 \rangle = \tfrac{1}{2}\sigma^2 \langle |\mathrm{grad}\ V|^2 \rangle \tag{56}$$

$$\langle \dot{L}_{ij} \rangle = \langle \{L_{ij}, H_0\} \rangle. \tag{57}$$

We can, therefore, conclude that the mean value of the deterministic energy increases with time, whereas if the angular momentum is conserved in the deterministic evolution, its mean value will also be conserved in the perturbed system, even when the perturbation breaks down the rotational symmetry of the Hamiltonian H_0. Unfortunately it is not possible to obtain similar results for the coloured noise since the complexity of the expressions does not allow us to carry out such qualitative analysis.

4.2. The nonlinear Klein-Gordon equation

The nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + V_0'(u) = 0 \tag{58}$$

possesses a Hamiltonian structure [36]. Here the subscripts indicate partial derivatives with respect to the correspondent variable. The phase space is that of smooth functions $u(x)$, $\pi(x)$ on $\mathbb{R}$ and a Poisson bracket is defined on the functionals $A[u, \pi]$ by

$$\{A, B\} \equiv \int_{\mathbb{R}} \mathrm{d}x \left(\frac{\delta A}{\delta u}\frac{\delta B}{\delta \pi} - \frac{\delta A}{\delta \pi}\frac{\delta B}{\delta u} \right). \tag{59}$$

The Hamiltonian

$$H_0 = \tfrac{1}{2} \int_{\mathbb{R}} \mathrm{d}x (\pi^2 + u_x^2 + 2V_0) \tag{60}$$

with the Poisson bracket defined above, yields the evolution equations

$$u_t = \pi \tag{61}$$

$$\pi_t = u_{xx} - V_0'(u) \tag{62}$$

which coincide with (58).

We now consider the stochastic equation [37, 38]

$$u_{tt} - u_{xx} + V_0'(u) + \xi(t) V'(u) = 0 \tag{63}$$

which is derived from the stochastic Hamiltonian

$$H = H_0 + \xi(t) \int_{\mathbb{R}} \mathrm{d}x\, V(u(x)). \tag{64}$$

Recalling that, if

$$A[u] = \int_{\mathbb{R}} \mathrm{d}x\, A(u(x), u_x(x), u_{xx}(x), \ldots) \tag{65}$$

the functional derivative is

$$\frac{\delta A}{\delta u} = \frac{\partial A}{\partial u} - \left(\frac{\partial A}{\partial u_x}\right)_x + \left(\frac{\partial A}{\partial u_{xx}}\right)_{xx} - \ldots \tag{66}$$

it is easy to calculate the BFPA of the functional W up to second order on τ for the OU case

$$W = \frac{\sigma^2}{2}\left\{\int_{\mathbb{R}} \mathrm{d}x\, V - \tau \int_{\mathbb{R}} \mathrm{d}x\, V' u_t + \tau^2 \int_{\mathbb{R}} \mathrm{d}x [V'' u_t^2 + V'(u_{xx} - V_0')]\right\}. \tag{67}$$

Particularising (39) for u and π we obtain

$$\langle u \rangle_{tt} - \langle u \rangle_{xx} + \langle V_{\mathrm{ef}}'(u) \rangle + \langle \alpha^2(u) u_t \rangle = 0 \tag{68}$$

with

$$V_{\mathrm{ef}}(u) = V_0(u) - \frac{\tau\sigma^2}{4} [V'(u)]^2 \tag{69}$$

and

$$\alpha(u) = \tau\sigma V''(u). \tag{70}$$

If the potential V is quadratic on u, the last term of (68), due to the non-zero correlation time of the perturbation, is dissipative as in the Newtonian case.

The Klein-Gordon momentum functional

$$P = \int_{\mathbb{R}} \mathrm{d}x\, \pi u_x \tag{71}$$

is a conserved quantity in the deterministic evolution. Although we do not have the explicit form of W, it is possible to calculate the evolution equation for a quantity $\langle B \rangle$ in all orders of τ if B and $\{B, V\}$ are constants of the deterministic motion

$$\begin{aligned}\{B, W\} &= \int_0^\infty \mathrm{d}s\, \Gamma(s) \{B, [\phi_0^{-s}]^* V\} \\ &= \int_0^\infty \mathrm{d}s\, \Gamma(s) [\phi_0^{-s}]^* \{B, V\} \\ &= \frac{\sigma^2}{2} \{B, V\}.\end{aligned} \tag{72}$$

This is the case for $\langle P \rangle$ since

$$\begin{aligned}A[u] \equiv \{P, V\} &= -\int_{\mathbb{R}} \mathrm{d}x\, u_x V'(u) \\ &= V(u(-\infty)) - V(u(+\infty))\end{aligned} \tag{73}$$

and the functional derivative of $A[u]$ is identically zero. Note that if the solution $u(x)$ does not vanish at infinity the functional $A[u]$ can be non-zero as it happens in the kink and antikink solutions of the sine-Gordon equation [37, 38]. We can now also apply (72) to $\langle P^2\rangle$ since

$$\{P^2, V\} = 2P\{P, V\} = 2PA \tag{74}$$

is a constant of the deterministic motion. We have

$$\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}t} = 0 \tag{75}$$

$$\frac{\mathrm{d}\langle P^2\rangle}{\mathrm{d}t} = \sigma^2\langle (A[u])^2\rangle \tag{76}$$

which extends to coloured perturbation the result obtained for the white noise case in [37].

The mean value of the deterministic energy verifies

$$\begin{aligned}\langle \dot{H}_0\rangle = \frac{\sigma^2}{2}\int_{\mathbb{R}} \mathrm{d}x\langle [V'(u)]^2\rangle - \int_{\mathbb{R}} \mathrm{d}x\langle \alpha^2(u)(u_t^2 + u_x^2)\rangle \\ + \frac{\sigma^2\tau^2}{2}\int_{\mathbb{R}} \mathrm{d}x\langle V'(u)[(u_t^2 - u_x^2)V'''(u) - V_0''] - V''V_0'\rangle.\end{aligned} \tag{77}$$

From this expression we see that for white noise perturbations ($\tau = 0$) the mean value of the energy has the same behaviour as for Newtonian systems, i.e., it increases with time.

4.3. *The Korteweg–de Vries equation*

The KdV equation [39]

$$u_t + u_{xxx} - 6uu_x = 0 \tag{78}$$

can be discussed in a similar way. The phase space is now the set of functions $u(x)$ and one of the possible Hamiltonian structures is given by the Poisson bracket

$$\{A, B\} = \int_{\mathbb{R}} \mathrm{d}x \frac{\delta A}{\delta u}\left(\frac{\delta B}{\delta u}\right)_x \tag{79}$$

with the Hamiltonian functional

$$H_0(u) = \int_{\mathbb{R}} \mathrm{d}x(u^3 + \tfrac{1}{2}u_x^2). \tag{80}$$

As in the preceding subsection we now introduce a stochastic perturbation, coupled to the potential $V(u)$, through the fluctuating Hamiltonian $H = H_0 + \xi(t)\int_{\mathbb{R}} \mathrm{d}x\, V(u(x))$.

The stochastic evolution equation is

$$u_t + u_{xxx} - 6uu_x - \xi(t)V''(u)u_x = 0. \tag{81}$$

For simplicity we limit the discussion to the white noise case. The equation for the mean value of the field reads

$$\langle u\rangle_t + \langle u\rangle_{xxx} - 6\langle uu_x\rangle = 0. \tag{82}$$

To find the behaviour of the mean value of the main relevant functionals of the problem, i.e. the mass

$$M \equiv \int_{\mathbb{R}} \mathrm{d}x\, u(x) \tag{83}$$

the momentum

$$P \equiv \int_{\mathbb{R}} \mathrm{d}x\, u^2(x) \tag{84}$$

and the energy (80), we use the general expression (8). All these functionals are constants of the deterministic motion, and it is not difficult to see that the mean value of the mass and momentum is constant in the stochastic evolution equation, and that this also occurs for coloured noise. On the other hand, the mean value of the deterministic energy increases with time verifying

$$\langle \dot{H}_0 \rangle = \frac{\sigma^2}{2} \int_{\mathbb{R}} \mathrm{d}x\, \langle [V''' u_x^2]^2 \rangle. \tag{85}$$

This mean value is constant if $V'''(u) = 0$. In this case the equation can be exactly solved using the inverse spectral transformation [38] and has soliton solutions.

5. Quantum systems perturbed by classical noise

The linear symplectic structure of quantum mechanics [36] allows us to apply the expression (39) to Schrödinger equations with classical stochastic perturbations. We consider the quantum stochastic Hamiltonian

$$H = H_0 + \xi_k(t) V_k \tag{86}$$

where $\xi_k(t)$ are real classical stochastic processes as considered in the previous sections and H_0, V_k are self-adjoint operators on the Hilbert space of the system. Using (39) and the Poisson bracket of the symplectic structure it is possible to obtain, for any observable A_t in the Heisenberg picture, the evolution equation

$$\langle \dot{A}_t \rangle = -\mathrm{i} \langle [A, H_0]_t \rangle - \langle [[A, V_k], W_k(t)]_t \rangle \tag{87}$$

where, in the BFPA,

$$W_k(t) = \sum_{k'} \int_0^t \mathrm{d}s\, \Gamma_{kk'}(s)\, \mathrm{e}^{-\mathrm{i}sH_0} V_{k'}\, \mathrm{e}^{\mathrm{i}sH_0}. \tag{88}$$

In the basis of eigenvectors $|\psi_i\rangle$ of H_0, W_k can be written as

$$\langle \psi_i | W_k(t) | \psi_j \rangle = \sum_{k'} \int_0^t \mathrm{d}s\, \Gamma_{kk'}(s)\, \mathrm{e}^{-\mathrm{i}s\omega_{ij}} \langle \psi_i | V_{k'} | \psi_j \rangle \tag{89}$$

where $\omega_{ij} = \varepsilon_i - \varepsilon_j$, ε_i being the eigenvalue of H_0 corresponding to the eigenvector $|\psi_i\rangle$. If the noises are of the OU type and the transient terms are neglected we have

$$\langle \psi_i | W_k | \psi_j \rangle = \sum_{k'} \frac{\sigma_{kk'}^2}{2[1 + \mathrm{i}\tau_{kk'}\omega_{ij}]} \langle \psi_i | V_{k'} | \psi_j \rangle. \tag{90}$$

Another expression for W_k is given in terms of a τ expansion:

$$W_k = \sum_{k'} \frac{\sigma^2_{kk'}}{2} \sum_{n=0}^{\infty} \tau^n_{kk'}(-\mathrm{i\,ad}_{H_0})^n(V_{k'}) \tag{91}$$

with

$$\mathrm{ad}_H(V) \equiv [H, V]. \tag{92}$$

If the initial state of the system is ρ_0, the density matrix ρ_t defined by

$$\mathrm{Tr}[\rho_t A] = \mathrm{Tr}[\rho_0 \langle A_t \rangle] \tag{93}$$

follows the evolution equation

$$\dot{\rho}_t = -\mathrm{i}[H_0, \rho_t] - [V_k, [W_k, \rho_t]]. \tag{94}$$

This equation reduces in the white noise case to the one derived in [40] and generates a quantum semigroup [41].

We will now discuss as an example of the BFPE the depolarisation of a spin $\boldsymbol{L}$ in the presence of a magnetic field $\boldsymbol{B} = B\boldsymbol{k}$ with coloured fluctuations, which has been discussed as a model to explain the radiative correction to the magnetic moment [30] and, very recently, in the problem of spectral line-shape broadening [31]. To simplify our analysis we restrict consideration to the case of isotropic fluctuations, although the extension to the non-isotropic situation is straightforward.

The Hilbert space $\mathscr{H}$ of such system is a $(2l+1)$-dimensional complex space and its Hamiltonian is

$$H = BL_3 + \xi_1(t)L_1 + \xi_2(t)L_2 + \xi_3(t)L_3 \tag{95}$$

where ξ_k are independent OU random processes with the same dispersion σ and correlation time τ.

The deterministic evolution given by the unitary operator $\mathrm{e}^{-\mathrm{i}sH_0}$ is a rotation around the z axis with angle Bs. Trivially $W_3 = (\sigma^2/2)L_3$. We also have

$$\begin{aligned} W_1 &= \frac{\sigma^2}{2\tau}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s/\tau}[L_1\cos(-Bs) - L_2\sin(-Bs)] \\ &= \frac{\sigma^2}{2}\frac{L_1 + \tau BL_2}{1+(\tau B)^2}. \end{aligned} \tag{96}$$

A similar calculation gives

$$W_2 = \frac{\sigma^2}{2}\frac{L_2 - \tau BL_1}{1+(\tau B)^2}. \tag{97}$$

Using the Jacobi identity it is possible to write the equation for ρ in the form

$$\dot{\rho} = -\mathrm{i}\hat{B}\,\mathrm{ad}_{L_3}(\rho) - \frac{\sigma^2}{2(1+B^2\tau^2)}\left(\sum_{j=1}^{3}(\mathrm{ad}_{L_j})^2(\rho) + B^2\tau^2(\mathrm{ad}_{L_3})^2(\rho)\right) \tag{98}$$

with

$$\hat{B} = B\left(1 + \frac{\tau\sigma^2}{2(1+(\tau B)^2)}\right). \tag{99}$$

Although in this example we have considered coloured noises the equation (98) is the generator of a quantum semigroup. Recalling that ad_{L_j} is a reducible self-adjoint representation of the Lie algebra su(2) on the Hilbert space of the Hilbert-Schmidt operators on $\mathscr{H}$, we can solve (98) using the angular momentum technique. The operators ad_{L_3} and $\Sigma_{j=1}^{3}(\mathrm{ad}_{L_j})^2$, defined on the representation space, commute and are self-adjoints. Therefore, they have a common basis of eigenvectors $|k\,m\rangle$ which verifies

$$\mathrm{ad}_{L_3}(|k\,m\rangle) = m|k\,m\rangle \qquad \sum_{j=1}^{3} (\mathrm{ad}_{L_j})^2(|k\,m\rangle) = k(k+1)|k\,m\rangle \tag{100}$$

with $m=-k,\ldots,k$ and $k=0,1,\ldots,2l$. There also exist two ladder operators $\mathrm{ad}_{L_\pm} \equiv \mathrm{ad}_{L_1} \pm \mathrm{i}\,\mathrm{ad}_{L_2}$ which act on that basis in the form

$$\mathrm{ad}_{L_\pm}(|k\,m\rangle) = \sqrt{k(k+1)-m(m\pm 1)}\,|k\,m\pm 1\rangle. \tag{101}$$

Every eigenvector $|k\,m\rangle$ can be obtained from these operators using (101) and the fact that $L_\pm^k \propto |k\ \pm k\rangle$. Therefore, if we write $\rho(t)$ in this basis

$$\rho(t) = \sum \alpha_{km}(t)|k\,m\rangle \tag{102}$$

and we introduce it in (98), we obtain for the coefficients $\alpha_{km}(t)$

$$\dot{\alpha}_{km}(t) = \left(-\mathrm{i}\hat{B}m - \frac{\sigma^2}{2(1+B^2\tau^2)}\,(k(k+1)+B^2\tau^2 m^2)\right)\alpha_{km}(t). \tag{103}$$

We note that all the coefficients vanish exponentially except for $\mathbf{1}=|0\,0\rangle$. The maximum of the relaxation times is

$$\tau_{\mathrm{rel}} = \frac{1+B^2\tau^2}{\sigma^2} \tag{104}$$

and the maximum entropy state $1/(2l+1)\mathbf{1}$ is reached for every initial condition, in an exponential way with relaxation time given by the last expression.

The mean value of the components of the angular momentum are easily obtained using (87) or by calculating the trace of $\rho(t)L_i$ using (102) and (103) and taking into account that the basis $|k\,m\rangle$ is orthonormal in the scalar product $\langle A|B\rangle = \mathrm{Tr}(A^+B)$. Either way we get the polarisation vector

$$P_1(t) = [P_1(0)\cos(\hat{B}t) + P_2(0)\sin(\hat{B}t)]\exp\left[-\frac{t}{\tau_{\mathrm{rel}}}\left(1+\frac{B^2\tau^2}{2}\right)\right] \tag{105}$$

$$P_2(t) = [-P_1(0)\sin(\hat{B}t) + P_2(0)\cos(\hat{B}t)]\exp\left[-\frac{t}{\tau_{\mathrm{rel}}}\left(1+\frac{B^2\tau^2}{2}\right)\right] \tag{106}$$

$$P_3(t) = \exp\left[-\frac{t}{\tau_{\mathrm{rel}}}\right] \tag{107}$$

which coincides with the result obtained in [31] using the more complicated cumulant method.

Therefore, a depolarisation occurs, and the spin precesses with frequency $\hat{B}$ which is greater than the deterministic one. This increasing in the precession velocity is entirely due to the non-zero correlation time.

We would like to point out that it is a simple matter to extend the above discussion to arbitrary correlation function $\Gamma(s)$. Equation (103) now reads

$$\dot{\alpha}_{km}(t) = \left[-\mathrm{i}\hat{B}m - \Gamma_c(B)\left(k(k+1) + \frac{\Gamma_c(0)-\Gamma_c(B)}{\Gamma_c(B)}\,m^2\right)\right]\alpha_{km}(t) \tag{108}$$

and the precession frequency is now

$$\hat{B} = B + \Gamma_s(B) \tag{109}$$

where Γ_c and Γ_s are the Fourier cosine and sine transform, respectively

$$\Gamma_c(\omega) = \int_0^\infty ds\, \Gamma(s) \cos \omega s \tag{110}$$

$$\Gamma_s(\omega) = \int_0^\infty ds\, \Gamma(s) \sin \omega s. \tag{111}$$

The cosine transform is proportional to the spectral density of the processes $\xi_k(t)$ [1]. Therefore $\Gamma_c(0) \geqslant \Gamma_c(B) \geqslant 0$ and we always have depolarisation with a maximum relaxation time given by

$$\tau_{rel} = \frac{1}{2\Gamma_c(B)}. \tag{112}$$

On the other hand, since

$$\frac{1}{\Gamma_c(B)} \geqslant \frac{1}{\Gamma_c(0)} \tag{113}$$

and $\Gamma_c(0)$ is the intensity of the noise, the use of the equations in the stationary regime is justified for sufficiently weak noises and short correlation time.

We finally mention that the results in [30] are altogether recovered if we retain only the linear term of $\Gamma_s(B)$ in (109).

6. Conclusions

In this paper we have shown how the geometrical language can be useful to analyse the behaviour of nonlinear systems perturbed by coloured noise. Within this framework we are able to distinguish different assumptions that lead to various effective Fokker-Planck equations. The geometrical approach is particularly adequate to study Hamiltonian systems in both the classical and quantum cases, and seems very promising for stochastic partial differential equations. In particular we obtain relevant information for the evolution of some dynamical variables for the nonlinear Klein-Gordon equation (which includes the well known sine-Gordon and ϕ^4), the Korteweg-de Vries equation, and the behaviour of a spin in a fluctuating magnetic field, generalising previous results.

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Appendix

In this appendix we calculate the mean value $\langle \xi_k(t) B_t \rangle$ for an arbitrary dynamical variable B. In order to do this we use Novikov's theorem [21]

$$\langle \xi_k(t)\mathscr{R}[\xi]\rangle = \sum_j \int_0^t \mathrm{d}t' \langle \xi_k(t)\xi_j(t')\rangle \left\langle \frac{\delta \mathscr{R}[\xi]}{\delta \xi_j(t')} \right\rangle \tag{114}$$

which is valid for every functional $\mathscr{R}[\xi]$ of the stochastic processes $\xi_k(t)$.

Introducing B_t in the last expression

$$\langle \xi_k(t)B_t\rangle = \sum_{k'} \int_0^t \Gamma_{kk'}(t-t') \left\langle \frac{\delta B_t}{\delta \xi_{k'}(t')} \right\rangle. \tag{115}$$

We integrate the evolution equation (5) to evaluate the functional derivative

$$B_t = B_0 + \int_0^t \mathrm{d}s\{[L_F(B)]_s + \xi_k(s)[L_{G^k}(B)]_s\}. \tag{116}$$

Taking the functional derivative

$$\frac{\delta B_t}{\delta \xi_k(t')} = \int_{t'}^t \mathrm{d}s \left(\frac{\delta [L_F(B)]_s}{\delta \xi_k(t')} + \xi_{k'}(s) \frac{\delta [L_{G^{k'}}(B)]_s}{\delta \xi_k(t')} \right) + [L_{G^k}(B)]_{t'} \tag{117}$$

where we have taken into account that the dynamical variables evaluated at s depend on $\xi_k(t')$ only for $s \geqslant t'$.

There exists a field $\boldsymbol{D}^k(t, t')$ such that

$$L_{\boldsymbol{D}^k(t,t')}(B_t) = \frac{\delta B_t}{\delta \xi_k(t')}. \tag{118}$$

From (117) it is easy to see that $\boldsymbol{D}^k(t', t') = \boldsymbol{G}^k_{t'}$. Introducing this field in (117) and deriving with respect to t

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\boldsymbol{D}^k(t,t')}(B_t) = L_{\boldsymbol{D}^k(t,t')}[L_{F_t}(B_t) + \xi_{k'}(t) L_{G^{k'}_t}](B_t). \tag{119}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\boldsymbol{D}^k(t,t')}(B_t) = L_{(\mathrm{d}/\mathrm{d}t)\boldsymbol{D}^k(t,t')}(B_t) + L_{\boldsymbol{D}^k(t,t')}\left(\frac{\mathrm{d}}{\mathrm{d}t} B_t\right) \tag{120}$$

hence, using (5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{D}^k(t, t') = 0 \tag{121}$$

and finally

$$\boldsymbol{D}^k(t, t') = \boldsymbol{G}^k_{t'}. \tag{122}$$

The mean value can be written

$$\begin{aligned} \langle \xi_k(t)B_t\rangle &= \sum_{k'} \int_0^t \mathrm{d}t' \, \Gamma_{kk'}(t-t') \langle L_{\boldsymbol{D}^k(t,t')}(B_t)\rangle \\ &= \langle [L_{\boldsymbol{D}^k(t)}(B)]_t \rangle \end{aligned} \tag{123}$$

where

$$\begin{aligned} \boldsymbol{D}^k(t) &= \sum_{k'} \int_0^t \mathrm{d}t' \, \Gamma_{kk'}(t-t')[\phi^{(0,t)}]^*[\phi^{(t',0)}]^* \boldsymbol{G}^{k'} \\ &= \sum_{k'} \int_0^t \mathrm{d}t' \, \Gamma_{kk'}(t-t')[\phi^{(t',t)}]^* \boldsymbol{G}^{k'}. \end{aligned} \tag{124}$$

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